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OSCULATORY INTERPOLATION FORMULAS.

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The best formulas of interpolation, making use of finite differences, are Newton's, Everett's, Stirling's, and Bessel's which are as follows:

Newton's formula:

$$u_x = u_0 + x\Delta u_0 + \frac{x(x-1)}{2!}\Delta^2 u_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 u_0 + \cdots.$$

Everett's formula:

$$u_x = f(\xi)u_0 + f(x)u_1$$

$$\text{where } f(x) = x + \frac{(x+1)^{(3)}}{3!}\delta^2 + \frac{(x+2)^{(5)}}{5!}\delta^4$$

$$= \left(x + \frac{(x+1)x(x-1)}{3!}\delta^2 + \frac{(x+2)(x+1)x(x-1)(x-2)}{5!}\delta^4\right)$$

and ξ is the distance of u_x measured behind u_1.

Stirling's formula:

$$u_x = u_0 + n\delta u_0 + \frac{n^2}{2!}\delta^2 u_0 + \frac{n(n^2-1)}{3!}\bar{\delta}^3 u_0 + \frac{n^2(n^2-1)}{4!}\delta^4 u_0$$
$$+ \frac{n(n^2-1)(n^2-4)}{5!}\bar{\delta}^5 u_0 + \cdots.$$

Bessel's formula:

$$u_x = u_0 + n\delta u_{\frac{1}{2}} + \frac{n(n-1)}{2!}\bar{\delta}^2 u_{\frac{1}{2}} + \frac{n(n-1)(n-\frac{1}{2})}{3!}\delta^3 u_{\frac{1}{2}}$$
$$+ \frac{n(n^2-1)(n-2)}{4!}\bar{\delta}^4 u_{\frac{1}{2}} + \frac{n(n^2-1)(n-2)(n-\frac{1}{2})}{5!}\delta^5 u_{\frac{1}{2}} + \cdots.$$

Thus Newton's formula is based upon ordinary differences and the three others upon central differences whose notation is explained by the following table:

$$
\begin{array}{llllll}
u_{-2} & & & & & \\
 & \delta u_{-\frac{3}{2}} & & & & \\
u_{-1} & & \delta^2 u_{-1} & & & \\
 & \delta u_{-\frac{1}{2}} & & \delta^3 u_{-\frac{1}{2}} & & \\
u_0 & & \delta^2 u_0 & & \delta^4 u_0 & \\
 & \delta u_{\frac{1}{2}} & & \delta^3 u_{\frac{1}{2}} & & \delta^5 u_{\frac{1}{2}} \\
u_1 & & \delta^2 u_1 & & \delta^4 u_1 & \\
 & \delta u_{\frac{3}{2}} & & \delta^3 u_{\frac{3}{2}} & & \\
u_2 & & \delta^2 u_2 & & & \\
 & \delta u_{\frac{5}{2}} & & & & \\
u_3 & & & & &
\end{array}
$$

We shall use the central difference symbols but with a dash over them to designate the arithmetic averages of the differences of the same order on both sides of them. Thus $\bar{\delta} u_0 = \frac{1}{2}(\delta u_{\frac{1}{2}} + \delta u_{-\frac{1}{2}})$, and so on for the rest. We believe that this latter idea is an improvement over that of Mr. Sheppard,* who contrived the scheme of representing central differences shown in the above table, in which a separate character μ is introduced instead of the dash which we suggest. We consider the use of the character μ in such a way to be very misleading.

The purpose of this paper is threefold. First we wish to give the results of our derivations of the osculatory formulas based upon Stirling's and Bessel's formulas corresponding to those which have been derived based upon Newton's† and Everett's‡ formulas, all to be used for interpolating four values between two of a series of equidistant values by fifth differences. Second, we wish to bring together all four osculatory formulas for purposes of comparison and future reference. Third, we wish to offer for future reference formulas for computing fifth leading differences to be used in connection with the applications of the four osculatory interpolation formulas, to simplify the computation when several successive intervals are to be considered.

As is well known, all interpolations within a series of successive intervals are performed through the use of a series of what might be called partial interpolation curves. In osculatory interpolation, the "joints" of these separate interpolation curves are welded together in a more satisfactory manner by requiring that not only the slopes but also the curvatures of any two interpolation curves to be the same at their point or points of intersection. Thus the two interpolation curves, one through the points u_{-2}, u_{-1}, u_0, u_1, and u_2, the other through u_{-1}, u_0, u_1, u_2, and u_3 are required to have equal slopes and equal curvatures at the two points u_0 and u_1 for purposes of interpolating any number of values (four, in this paper) between u_0 and u_1. This modification tends to remove the discontinuities usually found at the intersections of two distinct interpolation curves.

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* Proceedings of the London Mathematical Society, Vol. XXXI, p. 449.

† Journal of the Institute of Actuaries, Vol. 22., p. 270.

‡ J. I. A., Vol. 42, p. 369.

This method of interpolation has been termed "osculatory" interpolation because the successive pairs of partial interpolation curves are thus seen to have a common osculating circle at their points of intersection. It was first devised by Sprague* and later developed and applied by Karup,† King,‡ Buchanan,§ and others.‖ The method or rather modification has already been applied to Newton's* and Everett's§ formulas, the latter of which involves only even central differences of each of the middle pairs of the series of values between which the interpolations are to be made. The central difference formulas, Stirling's¶ and Bessel's, are so widely used that it seemed well worth while to obtain the osculatory formulas for them as well as for Newton's and Everett's formulas.

The process of deriving the osculatory formulas wherein analytically first derivatives are considered in lieu of slopes and second derivatives in lieu of curvatures has been explained so well in connection with the derivation of Sprague's formula (based upon Newton's formula) and Buchanan's formula (based upon Everett's formula), and especially in an independent note by Lidstone**, that we shall give simply the final equations of the two curves. The one based upon Stirling's formula becomes

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* Journal of the Institute of Actuaries, Vol. 22, p. 270.

† Trans. Second International Actuarial Congress, p. 78.

‡ J. I. A., Vol. 42, p. 225.

§ J. I. A., Vol. 42, p. 369.

‖ Quarterly Pub. Amer. Stat. Assoc., June, 1910.
The Record. Amer. Inst. of Actuaries, June, 1911.

¶ Text-book. Institute of Actuaries, p. 447.

** J. Inst. Act., Vol. 42, p. 394.

$$u_x = u_0 + x\bar{\delta}u_0 + \frac{x^2}{2}\delta^2u_0 + \frac{(x^3-x)}{3!}\bar{\delta}^3u_0 + \frac{(x^4-x^2)}{4!}\delta^4u_0 + \left(\frac{x^5}{20}-\frac{x^3}{4}+\frac{x}{5}\right)\frac{\bar{\delta}^5u_0}{3!} \tag{1}$$

The equation based upon Bessel's formula becomes

$$u_x = u_0 + x\delta u_{\frac{1}{2}} + \frac{(x^2-x)}{2}\bar{\delta}^2u_{\frac{1}{2}} + \left(x^3-\tfrac{3}{2}x^2+\frac{x}{2}\right)\frac{\delta^3u_{\frac{1}{2}}}{3!} + \frac{(x^4-2x^3-x^2+2x)}{4!}\bar{\delta}^4u_{\frac{1}{2}} + \frac{(-x^5+2x^4-x^3)}{4!}\delta^5u_{\frac{1}{2}} \tag{2}$$

If we substitute successively $x=\frac{0}{5}, \frac{1}{5}, \cdots, \frac{5}{5}$ in equations (1) and (2), and difference each of the sets of six results five times we obtain the two desired sets of leading differences for interpolating four values between u_0 and u_1.

The osculatory leading differences based upon Stirling's formula are:

$$\begin{array}{lccccccccc}
(1) & \frac{\bar{\delta}u_0}{5} & +\frac{1}{2} & \frac{\delta^2u_0}{5^2} & -4 & \frac{\bar{\delta}^3u_0}{5^3} & - & \frac{\delta^4u_0}{5^4} & + & \frac{\bar{\delta}^5u_0}{5^4} \\
(2) & & 1 & \text{``} & +1 & \text{``} & -\frac{3}{2} & \text{``} & +3 & \text{``} \\
(3) & & & & 1 & \text{``} & +\frac{3}{2} & \text{``} & -3 & \text{``} \\
(4) & & & & & & 1 & \text{``} & -2 & \text{``} \\
(5) & & & & & & & & 5 & \text{``}
\end{array} \tag{A}$$

The osculatory leading differences based upon Bessel's formula are:

$$\begin{array}{lccccccccc}
(1) & \frac{\delta u_{\frac{1}{2}}}{5} & -2 & \frac{\bar{\delta}^2u_{\frac{1}{2}}}{5^2} & + & \frac{\delta^3u_{\frac{1}{2}}}{5^3} & + & \frac{9\,\bar{\delta}^4u_{\frac{1}{2}}}{5^4} & -\frac{2}{15} & \frac{\delta^5u_{\frac{1}{2}}}{5^4} \\
(2) & & 1 & \text{``} & -\frac{3}{2} & \text{``} & -4 & \text{``} & -\frac{1}{3} & \text{``} \\
(3) & & & & 1 & \text{``} & -1 & \text{``} & +\frac{1}{2} & \text{``} \\
(4) & & & & & & 1 & \text{``} & +0 & \text{``} \\
(5) & & & & & & & & -1 & \text{``}
\end{array} \tag{B}$$

The osculatory leading differences derived by Buchanan, based upon Everett's formula are:

$$\begin{array}{lcrrcrr}
(1) & -\frac{u_0}{5} & -6\,\frac{\delta^2u_0}{5^3} & +8\,\frac{\delta^4u_0}{5^4} & +\frac{u_1}{5} & -4\,\frac{\delta^2u_1}{5^3} & +\,\frac{\delta^4u_1}{5^4} \\
(2) & & 4\ \text{“} & -7\ \text{“} & & +1\ \text{“} & +3\ \text{“} \\
(3) & & -1\ \text{“} & +2\ \text{“} & & +1\ \text{“} & -3\ \text{“} \\
(4) & & & 3\ \text{“} & & & -2\ \text{“} \\
(5) & & & -5\ \text{“} & & & +5\ \text{“}
\end{array} \tag{C}$$

The osculatory leading differences derived by Sprague, based upon Newton's formula are:

$$\begin{array}{lcrrrr}
(1) & \frac{\Delta u_0}{5} & +8\,\frac{\Delta^2u_0}{5^2} & +11\,\frac{\Delta^3u_0}{5^3} & -11\,\frac{\Delta^4u_0}{5^4} & +\,\frac{\Delta^5u_0}{5^4} \\
(2) & & 1\ \text{“} & +6\ \text{“} & +1\ \text{“} & +3\ \text{“} \\
(3) & & & 1\ \text{“} & +4\ \text{“} & -3\ \text{“} \\
(4) & & & & 1\ \text{“} & -2\ \text{“} \\
(5) & & & & & 5\ \text{“}
\end{array} \tag{D}$$

To apply these different sets of leading differences, the observed series of values are differenced as usual, the differences averaged wherever necessary in connection with the central differences and finally each difference divided by the appropriate power of 5 (or multiplied by the corresponding decimal). The leading differences themselves are then formed according to the above tables. Each group of six original values will have its own set of differences for interpolating four values between the inmost pair, but if a series of successive intervals are to be interpolated the best plan is to interpolate the first interval as outlined above and then compute merely the appropriate fifth leading differences for the succeeding intervals such that by continued summation or addition to the preceding leading differences of lower order (in each case) the required interpolations will be obtained. The possibility of this plan has been pointed out before* and we shall merely carry it out as Buchanan has done in connection with his treatment of Everett's formula. Not being able to find the formula for computing these fifth differences in connection with Sprague's formula, we have derived these formulas for this case as well as for the osculatory formulas based upon

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*J. I. A. Vol. 42, p. 285.

Stirling's and Bessel's and derived in this paper. The method of derivation is outlined by Buchanan in his paper. These formulas or corrections of fifth differences for all four osculatory formulas are as follows:

	Newton's		Everett's		Stirling's		Bessel's	
	$\frac{\Delta^5 u_0}{5^4}$	$\frac{\Delta^6 u_0}{5^4}$	$\frac{\delta^5 u_{\frac{1}{2}}}{5^4}$	$\frac{\delta^5 u_{\frac{3}{2}}}{5^4}$	$\frac{\delta^5 u_{\frac{1}{2}}}{5^4}$	$\frac{\delta^5 u_1}{5^4}$	$\frac{\delta^5 u_{\frac{1}{2}}}{5^4}$	$\frac{\delta^6 u_1}{5^4}$
(1)	$+4$	1	3	1	3	1	$-\frac{1}{3}$	$4\frac{11}{30}$
(2)	-6		-6		-6		3	$-15\frac{13}{30}$
(3)	-6	-6		-6		-6	$\frac{1}{2}$	$17\frac{23}{30}$
(4)	4	3	1	3	1	3	$-1\frac{1}{6}$	$-6\frac{1}{5}$
(5)	5	5		5		5	-1	-1

The greatest disadvantage found in connection with the use of the above fifth differences is that usually the work should be carried to a greater number of decimals than if the above leading differences were used. If an appropriate computing machine is available this criticism is of no importance. As a general rule the work in connection with the leading differences should be carried to two more decimals than are to be finally retained and the work of the fifth differences to three or the maximum four.

One great advantage in the use of the above osculatory formulas is the accompanying check upon the computation, for the sets of leading differences not only lead to interpolations of the four values between a pair of intermediate original values but will also reproduce these original values and since the process of continued summation causes any errors in the computation to accumulate, such errors are readily caught.

Without going into the details, it should perhaps be pointed out that Bessel's and Everett's and hence the corresponding osculatory formulas lead to about the same degree of accuracy but are generally considered to be the most accurate of the four formulas considered in this paper. However, Everett's formula and the corresponding osculatory formula involve considerably less as well as simpler computation than Bessel's formula and its corresponding osculatory formula. The osculatory formula based upon Stirling's formula is very little less accurate than those based upon Everett's and Bessel's formulas and is remarkably easy to apply although scarcely as easy as Buchanan's (that based upon Everett's). There is scarcely any justification in the further use of Sprague's osculatory formula since the derivation of the formulas for the other osculatory interpolations, leading as it does to results nowise as accurate and involving much computation.

In conclusion the following example is given to show the use of the above sets of leading differences. The following values of u are arbitrarily chosen and the corresponding differences found in the usual way as shown.

$u_{-2}=4673423$					
	13462				
$u_{-1}=4686885$		6321			
	19783		384		
$u_0=4706668$		6705		65	
	26488		449		13
$u_1=4733156$		7154		78	
	33642		527		
$u_2=4766798$		7681			
	41323				
$u_3=4809121$					

Here we have, for example, $\bar{\delta} u_0=\frac{1}{2}(19783+26488)$

The results after applying the four sets of leading differences are:

By (A)	By (B)	By (C)	By (D)	(C)-(A)	(C)-(B)	(C)-(D)
$u_0=4706668$	4706668	4706668	4706668	0	0	0
4711406	4711397	4711397	4711416	9	0	19
4716418	4716401	4716401	4716435	17	0	34
4721710	4721691	4721691	4721729	19	0	38
4727287	4727273	4727273	4727301	14	0	28
$u_1=4733156$	4733156	4733156	4733156	0	0	0